Spinons in the Haldane–Shastry model: an ideal gas of half-fermions

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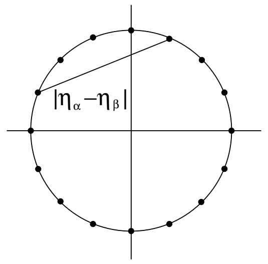
1 The Haldane–Shastry model

N sites with $\frac{1}{2}$ spins on the unit circle

$$\eta_\alpha = \exp\left(\frac{2\pi i}{N}\alpha\right), \quad \alpha = 1,\dots,N$$



Hamiltonian:

$$H_{\text{HS}} = J\left(\frac{2\pi}{N}\right)^2 \sum_{\substack{\alpha,\beta=1\\ \alpha<\beta}}^{N} \frac{\vec{S}_\alpha\cdot\vec{S}_\beta}{|\eta_\alpha-\eta_\beta|^2}$$

Symmetry generators:

$$\vec{S} = \sum_{\alpha=1}^{N} \vec{S}_\alpha\,, \quad \vec{\Lambda} = \frac{i}{2}\sum_{\substack{\alpha,\beta=1\\ \alpha\neq\beta}}^{N} \frac{\eta_\alpha+\eta_\beta}{\eta_\alpha-\eta_\beta}\left(\vec{S}_\alpha\times\vec{S}_\beta\right)$$

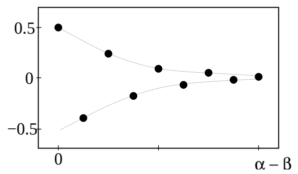
generate the Yangian,an associative, infinite dimensional algebra
$\Rightarrow$ infinite number of conserved quantities $\Rightarrow$ integrable system

2 Ground state

$$\Psi_0(z_1,\dots,z_M) = \prod_{\substack{i,j=1\\ i<j}}^{M}(z_i-z_j)^2\prod_{k=1}^{M} z_k\,, \quad E_0 = -J\frac{\pi^2}{24}\left(N+\frac{5}{N}\right)$$

for N even, $M=\frac{N}{2}$, with the z_i's the coordinates of the up spins

$|\Psi_0\rangle$ is a spin singlet, non degenerate and represents a spin liquid

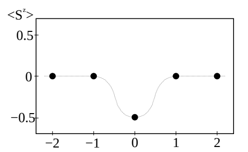


3 One-spinon states

$$\Psi_\alpha = \prod_{j=1}^{M}(\eta_\alpha - z_j)\,\Psi_0$$

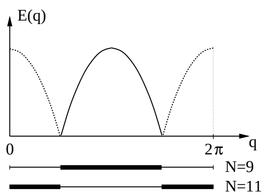
for N odd, $M=\frac{N-1}{2}$, localized spinon at site η_α

spin $\frac{1}{2}$, charge 0 $\rightarrow$ fractional quantization



Energy eigenstates:

$$\Psi_n = \frac{1}{N}\sum_{\alpha=1}^{N}(\eta_\alpha^*)^n\,\Psi_\alpha\,, \quad 0\le n\le M$$



Dispersion relation:

$$E(q) = \frac{J}{2}\left(\frac{\pi^2}{4}-q^2\right), \quad q_n = \frac{\pi}{2}N - \frac{2\pi}{N}\left(n+\frac{1}{4}\right)$$

4 Two-spinon states

For N even, $M=\frac{N-2}{2}$

$$\Psi_{\alpha\beta} = \prod_{j=1}^{M}(\eta_\alpha-z_j)(\eta_\beta-z_j)\,\Psi_0$$

represents two localized $\downarrow$ spinons in a triplet configuration.
Momentum eigenstates:

$$\Psi_{mn} = \frac{1}{N^2}\sum_{\alpha,\beta=1}^{N}(\eta_\alpha^*)^m(\eta_\beta^*)^n\Psi_{\alpha\beta}\,, \quad 0\le n\le m\le M$$

$$H_{\text{HS}}\,|\Psi_{mn}\rangle = E_{mn}\,|\Psi_{mn}\rangle + \sum_{l=1}^{l_M} V_{mn}^{l}\,|\Psi_{m+l,n-l}\rangle \tag{1}$$

$$E_{mn} = -J\frac{\pi^2}{24}\left(N-\frac{19}{N}+\frac{24}{N^2}\right) + \frac{J}{2}\left(\frac{2\pi}{N}\right)^2\left[m\,(M-m)+n\,(M-n)-\frac{m-n}{2}\right]$$

$$V_{mn}^{l} = -\frac{J}{2}\left(\frac{2\pi}{N}\right)^2(m-n+2l) \qquad \text{“scattering”}$$

Problem: $|\Psi_{mn}\rangle$ are not orthogonal

V_{mn}^{l} "scatters" only to lower energies $\Rightarrow$ energy eigenstates are of the form

$$\Phi_{mn} = \sum_{l=0}^{l_M} a_l^{mn}\Psi_{m+l,n-l}\,, \quad a_0^{mn}=1 \tag{2}$$

(1) yields a recursion formula for a_l^{mn}

The term $\frac{m-n}{2}$ in E_{mn} has been interpreted by Bernevig, Giuliano, and Laughlin PRL **86**, 3392 (2001); PRB **64**, 24425 (2001) as a spinon-spinon attraction.

Problem: How to define the single-spinon momenta?

Answer:

$$q_m = \frac{\pi}{2} - \frac{2\pi}{N}\left(m+\frac{3}{4}\right), \quad q_n = \frac{\pi}{2}-\frac{2\pi}{N}\left(n+\frac{1}{4}\right)$$

Then the two-spinon energy is

$$E_{mn} = -J\frac{\pi^2}{24}\left(N+\frac{5}{N}-\frac{6}{N^2}\right) + E(q_m) + E(q_n) \tag{3}$$

- shift between q_m and q_n by one-half of a momentum spacing $\frac{2\pi}{N}$ reflects the half-fermi statistics of the spinons
- E_{mn} equals the ground-state energy E_0 up to finite-size corrections plus the kinetic energies of the two spinons
- (3) is correct for all N
- the coefficients a_l^{mn} can be obtained without using the Hamiltonian by demanding orthogonality of the states Φ_{mn}

5 Dynamical spin suszeptibility

Dynamical spin suszeptibility (DSS):

$$\chi_q(\omega) \equiv -\text{Im}\,\langle\Psi_0|\,S_{-q}^{+}\frac{1}{\omega-(H_{\text{HS}}-E_0)+i0}S_q^{-}\,|\Psi_0\rangle$$

with

$$S_q^{\pm} = \sum_{\alpha=1}^{N}\eta_\alpha^k S_\alpha^{\pm}\,, \quad q=\frac{2\pi k}{N}$$

Localized spinons in terms of energy eigenstates:

$$|\Psi_{\alpha\beta}\rangle = \sum_{m=0}^{M}\sum_{n=0}^{m}\eta_\alpha^m\,\eta_\beta^n\,p_{mn}\!\left(\frac{\eta_\alpha}{\eta_\beta}\right)|\Phi_{mn}\rangle$$

p_{mn} are found to be hypergeometric functions. If the spinons are on top of each other:

$$p_{mn}(1) = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma(m-n+1)}{\Gamma\left(m-n+\frac{1}{2}\right)} \overset{m-n\to\infty}{\sim} \sqrt{m-n} \tag{4}$$

For the evaluation of the DSS:

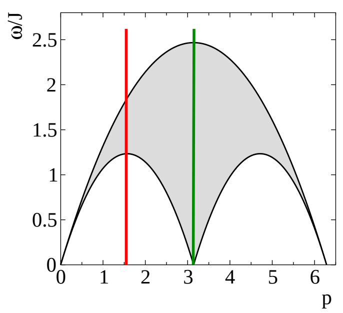
$$\begin{aligned} S_\alpha^{-}\,|\Psi_0\rangle &= \eta_\alpha\,|\Psi_{\alpha\alpha}\rangle \\ S_q^{-}\,|\Psi_0\rangle &= \sum_{\alpha=1}^{N}(\eta_\alpha)^k S_\alpha^{-}\,|\Psi_0\rangle \\ &= N\sum_{m=0}^{M}\sum_{n=0}^{m}\delta_{m+n+k+1,0}\,p_{mn}(1)\,|\Phi_{mn}\rangle \end{aligned}$$

DSS in the thermodynamic limit (Haldane and Zirnbauer PRL **71**, 4055 (1993)):

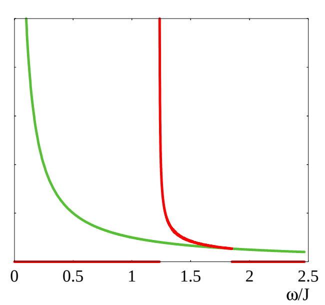
$$\chi_q(\omega) = \frac{J}{4}\frac{\Theta(\omega_2(q)-\omega)\;\Theta(\omega-\omega_{-1}(q))\;\Theta(\omega-\omega_1(q))}{\sqrt{\omega-\omega_{-1}(q)}\;\sqrt{\omega-\omega_1(q)}}$$

with the threshold energies $\omega_{-1},\omega_1,\omega_2$.

Two-spinon continuum:



Dynamical spin susceptibility:

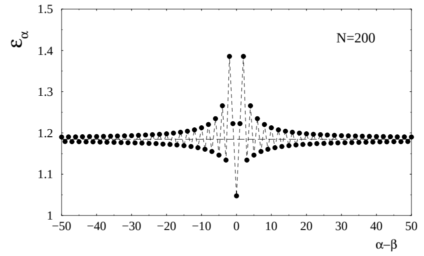


- local creation of two spinons creates predominatly spinons with lower energies, as is seen from (4) $\Rightarrow$ singularity at low energies
- singularity is due to the special structure of the spinon Hilbert space, not a result of the alleged spinon-spinon attraction
- square-root singularity is a generic feature of spin-$\frac{1}{2}$ chains

6 Energy of localized spinons

$$\mathcal{E}_{\alpha-\beta} \equiv \frac{\langle\Psi_{\alpha\beta}|H_{\text{HS}}|\Psi_{\alpha\beta}\rangle}{\langle\Psi_{\alpha\beta}|\Psi_{\alpha\beta}\rangle}$$

No energetic preference for small spinon separations
No spinon-spinon bound state



7 Asymptotic Bethe Ansatz

Bethe Ansatz equations for the pseudomomenta k_i:

$$k_i N = 2\pi I_i + \pi\sum_{j=1}^{M}\text{sign}(k_i-k_j)\,, \quad k_i\in[-\pi,\pi]$$

with the integer or half-integer quantum numbers I_i.
Energy and momentum of the states:

$$E = J\frac{\pi^2}{N^2}\frac{N(N^2-1)}{24} + \frac{J}{4}\sum_{i=1}^{M}(k_i^2-\pi^2)\,, \quad P=\sum_{i=1}^{M}(k_i+\pi).$$

In the thermodynamic limit introduce pseudomomenta density

$$\sigma(k_j) = \frac{1}{k_{j+1}-k_j}.$$

Solution for the ground state: $\sigma_0 = \frac{N}{4\pi}$.

Spinons are described by holes in the set $\{I_i\}$, i.e. a bare hole density

$$\sigma_{\text{h}}(k) = \sum_j \delta(k-\lambda_j).$$

The resulting pseudomomenta density is given by

$$\sigma(k) = \sigma_0(k) - \tfrac{1}{2}\sigma_{\text{h}}(k).$$

- no hole dressing
- one spinon reduces the number of available orbital by $\frac{1}{2}$
  $\Rightarrow$ spinons obey half-fermi statistics
- spinon-spinon scattering matrix (Eßler, PRB **51**, 13357 (1995))
  $$S=\pm i$$
  $\Rightarrow$ spinons in the Haldane–Shastry model do not interact

8 Conclusion

- the HS model is an integrable paradigm for a spin liquid
- the dynamical spin correlations show square-root singularity
- spinons in the Haldane–Shastry model are free